Modeling Timing Jitter in Oscillators

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Abstract

Timing jitter in oscillators is a key factor determining the phase noise performance of phase-locked loops. This paper reviews a theory of phase noise in oscillators, and presents timing jitter models suitable for discrete event simulation of these systems.

1. Introduction

Oscillators are used everywhere. They provide timing information that is needed to synchronize operations in electronic systems. A noise-free oscillator generates a periodically time-variant signal that is a perfect time reference. In reality, all oscillators exhibit phase noise and timing jitter. In Section 2, we review a theory of phase noise in oscillators. We then present two timing jitter models for oscillators used in conjunction with other digital circuits such as frequency dividers. The first model, given in Section 3, simulates time-domain jitter due to white noise. The second model, presented in Section 4, includes the effect of flicker noise. To the best of the author's knowledge, this is the first in which nonstationary and self-similar nature of flicker noise is modeled. These simulation models are useful for predicting phase noise and timing jitter in phase-locked loops, clock recovery circuits, and RF frequency synthesizers. We verify our results using a CMOS VCO circuit described in Section 5, and conclude the paper with a summary.

2. Theory of Noise in Oscillators

A noiseless oscillator provides a perfect time reference because the time-varying oscillator output at steady state, $x_s(t)$, is a T-periodic waveform, i.e. $x_s(t+T) = x_s(t)$, dividing time into equal lengths. This is depicted in Figure 1. Figure 2 shows what happens when the oscillator is perturbed by noise. Noise causes amplitude deviation $y(.)$ and phase deviation $f(t)$. We represent the noisy oscillator waveform $x(t)$ using the additive model:

$$x(s(t)) = x_s(s(t)) + y(s(t)) \text{ with } s(t) = t + \frac{T}{2p} f(t).$$

Phase deviation $f(t)$ naturally accumulates with time and drifts without bound, as oscillators are autonomous circuits. After the phase of an autonomous circuit has been perturbed, it persists and cannot be restored without information from other timing references. Amplitude deviation $y(s(t))$ always remains small and bounded due to the fact that nonlinear oscillators by design operate around very stable orbits, a bit like a person riding a roller coaster. Alternatively, we can define amplitude $a(.)$ as $a(s(t)) \equiv 1 + \dfrac{y(s(t))}{x_s(s(t))}$, provided the trajectory is not near a zero, and use the multiplicative model:

$$x(s(t)) = a(s(t))\, x_s(s(t)).$$

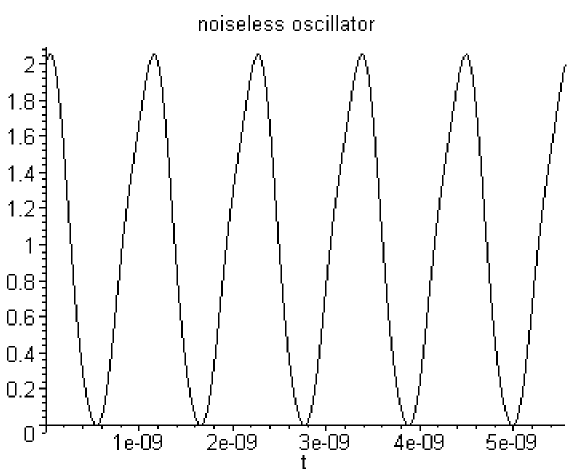


Figure 1. Periodic waveform of a noise-free oscillator

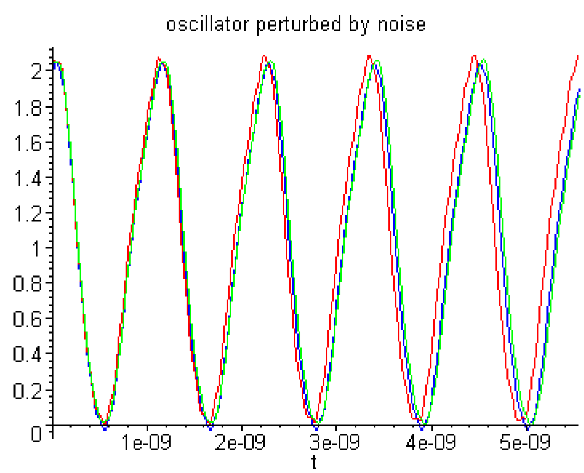


Figure 2. Waveforms of oscillator with phase and amplitude noise

We note that if there are no phase deviations, amplitude noise is periodically time-variant (T-cyclostationary), and has the same characteristics as noise found in a mixer that is periodically driven by a noise-free local oscillator. More important, due to the autonomous nature of oscillators, phase deviations cause "time"-shifts in $x_s(.)$ and in the amplitude process, $y(.)$ and $a(.)$. It is a common practice to ignore random time-drifts in the amplitude process, thereby introducing subtle modeling inconsistency.

Phase and amplitude deviation causes random variation in transition times and results in timing jitter that is depicted in Figure 3. Note the cumulative nature of phase errors with time. Timing jitter in oscillators can be attributed to $x_s(s(t))$ primarily, and to a smaller extent, $y(s(t))$ or $a(s(t))$ near the point of output transition. We first focus on the theory of oscillator phase noise, and then turn our attention to the modeling of timing jitter.

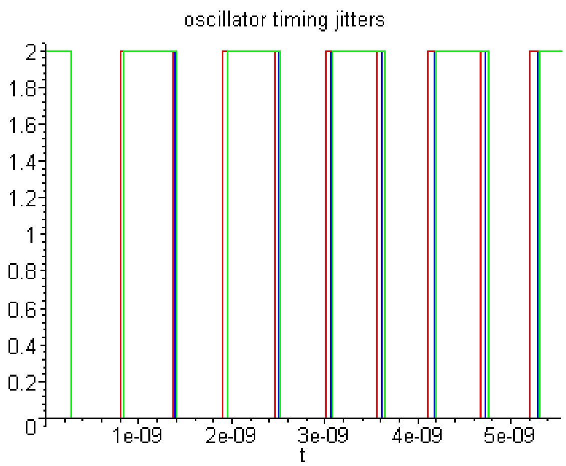


Figure 3. Oscillator timing jitter

We review several asymptotic results from the theoretical work of Kärtner[1] and Demir et. al.[2] for the case in which an oscillator is perturbed by white noise. First, the phase deviation $f(t)$ is a nonstationary[3] Wiener process, or Brownian motion. Second, $s(t)$ is Gaussian with variance that grows with time at a linear rate, say c. Third, the symmetric two-sided power spectral density[4] for the phase deviation is given by

$$Sf(f) = c\left(\frac{f_0}{f}\right)^2$$

where $f_0 = 1/T$ is the oscillation frequency. Lastly, the oscillator output $x(s(t))$ (and individually, $x_s(s(t))$ and $y(s(t))$) is a stationary process, and the power spectrum of $x_s(s(t))$ is the sum of lorentzians about the fundamental and its harmonics, as shown in Figure 4. In Figure 5, we plot the same spectrum at a positive frequency f_m offset from the fundamental. The power spectrum has no discrete impulses (spectral lines), as we would expect from an oscillator without a perfect frequency (time) reference. Phase noise causes spectral spreading of these impulses.

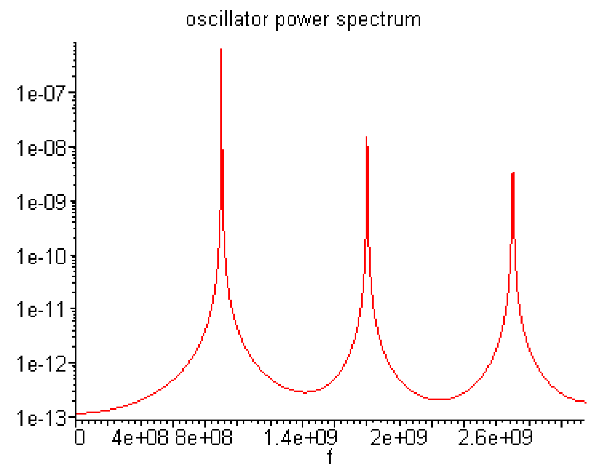


Figure 4. Oscillator power spectrum

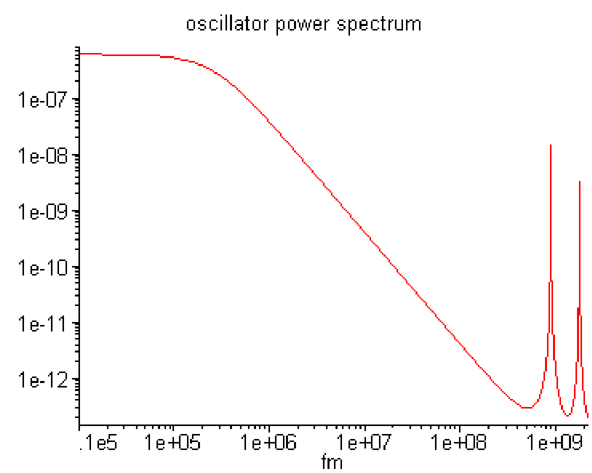


Figure 5. Logarithmic plot of oscillator power spectrum

A common metric for oscillator phase noise is the single sideband phase-noise spectrum, $L(f_m) = \dfrac{Noise}{Signal}$, a ratio with the spectral density of the noisy waveform $x_s(s(t))$ at frequency $f_0 + f_m$ in the numerator, and the discrete power spectrum of the ideal signal $x_s(t)$ at the fundamental frequency f_0 in the denominator. This power ratio has a mean-square spectrum given by a lorentzian with cutoff at $f_m = p\ f_0^2 c$, $\dfrac{f_0^2 c}{p^2 f_0^4 c^2 + f_m^2}$, or $L(f_m) \cong c\left(\dfrac{f_0}{f_m}\right)^2$ for frequencies above the corner frequency. Notice that the numerator has $x_s(s(t))$ that embeds a random $s(t)$ within a deterministic, "time"-varying $x_s(.)$. So this phase "noise" spectrum is quite different from the usual noise spectrum that characterizes just the time-averaged, additive amplitude noise riding on top of a deterministic signal in a non-autonomous circuit. We also note that the $L(f_m)$ spectrum is sometimes specified and measured using the spectral density of $x(s(t))$ and/or the total signal power. In contrast, the $Sf(f)$ spectrum is uniquely defined, independent of a signal, and is identical for all waveforms in an oscillator circuit. Another popular metric is the double sideband phase-noise spectrum, $2L(f_m)$. In Eldo RF, the reported $PHNOISE$ is $\sqrt{2L(f_m)}$. To extract c, we measure $PHNOISE$ at a frequency in the f_m^{-2} region where the slope of $DB(PHNOISE)$ is -20 dB/decade, and let c be $\dfrac{1}{2}\left(PHNOISE\,\dfrac{f_m}{f_0}\right)^2$. Alternatively, we equate $\dfrac{1}{2}(PHNOISE)^2$ to the lorentzian function, and let c be a solution to the quadratic equation. Once the parameter c is known, we can simulate the oscillator waveform using $x_s(s(t))$, where the random component of $s(t)$ is Brownian motion with variance growing at the linear rate c.

These results differ qualitatively and quantitatively from other works in several ways. In the theory proposed by Hajimiri and Lee[5], the phase deviation $f(t)$ is modeled as the time-integral of stationary noise $w(t)$ modulated by a deterministic, T-periodic "impulse sensitivity function" $p(t)$, i.e.

$$f'(t) = p(t)\ w(t).$$

The T-cyclostationary process, $p(t)\ w(t)$, is inconsistent with the fact that a free-running oscillator is not locked to a perfect phase reference, and is itself not a perfect time or frequency reference. We remark that the differential equation,

$$f'(t) = p(t + \frac{T}{2p} f(t))\ w(t),$$

includes random phase modulation of the T-periodic function, and is a self-consistent model even as noise $w(t)$ causes the phase deviation $f(t)$ to drift and become large, but we feel that it is not sufficiently compact and parsimonious for discrete event simulation.

In textbook analyses, the oscillator output spectrum is often determined using the superposition principle, a small-signal analysis, and a first-order additive model:

$$x_s(t + \frac{T}{2p} f(t)) \approx x_s(t) + x_s'(t)\frac{T}{2p} f(t).$$

The assumption that the phase deviation $f(t)$ is small is only valid over a short time interval. Such a simplified analysis predicts (incorrectly) that the output spectrum contains discrete impulses, and the oscillator output is a nonstationary process. There is a common misconception that $x_s'(t)\dfrac{T}{2p} f(t)$ (and the oscillator output) is cyclostationary, even though $f(t)$ is neither stationary nor cyclostationary. Over a short time interval, the phase variance remains small, and a single waveform is nearly periodic. Over a long time span, the phase variance grows large, a single waveform is not synchronous with itself over a widely separated time interval, and two oscillators cannot maintain synchronicity with a bounded phase difference. With the passage of time, $f(t) \bmod 2p$ for an ensemble of oscillators becomes randomized – ruling out a cyclostationary process, the ensemble statistics of the waveforms reach steady state and become stationary (independent of time). For instance, the ensemble average of $x_s(s(t))$ is asymptotically a constant, not the ideal $x_s(t)$ waveform. Time- and frequency-domain analysis techniques that fail to take into account the unbounded phase variance inherent in oscillators will make predictions that are inconsistent with the long-term behavior of oscillators and oscillatory systems such as frequency/phase locked loops.

Lastly, the phase noise spectrum $L(f_m)$ derived by Kundert[6] and others is in error by a factor of two, or -3.01 dBc/Hz, likely due to an inconsistent use of two-sided vs. one-sided quantities.

3. Timing Jitter Model

We now proceed to develop a timing jitter model for oscillators perturbed by white noise. We model $x_s(s(t))$ as

$$x_s(s(t)) = x_s\left(t + \sqrt{c}\int_0^t w(t)\, dt \right),$$

where $w(t)$ is stationary white noise. Discretization of $s(t)$ with a timestep Δt results in the discrete-time process defined by

$$s_{k+1} = \sum_{n=0}^{k} \Delta t + \sqrt{c\,\Delta t}\; w_n \;,$$

with w_n as independent, standard Gaussians. The second term is a random walk with stationary, independent increments whose variance is proportional to the timestep. We can simulate the oscillator waveform using the phase-modulated source $x_s(s_k)$ with a timestep that ensures each cycle has about 10 to 20 points. When the oscillator is used with digital circuits, we are interested in the output transition times, and not the detailed waveform. For level crossings at $x_s(k\,\Delta t)$, transition times are the first passage times $s^{-1}(k\,\Delta t)$. In the statistical literature, it is known that the inverse Gaussian distribution characterizes the first passage time in Brownian motion with linear drift. We notice that the Gaussian distribution of $s(k\,\Delta t)$ can approximate the transition time distribution for large $k\,\Delta t / c$. Therefore, we simulate the jittered transition times $s^{-1}(k\,\Delta t)$ of the oscillator using the discrete-time process s_k with $\Delta t = T/2$. The rms jitter for one cycle of oscillation (cycle-to-cycle jitter) is $\sqrt{cT}$.

4. Timing Jitter Model Including 1/f Noise

A typical oscillator phase-noise spectrum is of the form

$$L(f_m) \approx f_0^2 (c\, f_m^{-2} + c_{FN}\, f_m^{-3})$$

for frequencies above the corner frequency. This is in the same general form as Leeson's phenomenological model[7] without a flat region[8]. The $c\, f_m^{-2}$ term is due to white noise such as thermal noise and shot noise in devices. Flicker or 1/f noise in devices results in the $c_{FN}\, f_m^{-3}$ term, which is dominant at small frequency offsets. We now extend our timing jitter model to include the effect of nonstationary 1/f noise. One common method to generate noise with an arbitrary spectral density is to use a sum of sinusoids with random phases[9]. The simulated "noise" is a deterministic, smooth waveform. It has a discrete spectrum that samples the continuous spectral density. Another method is to synthesize a linear time-invariant network to filter and reshape white noise. For example, a causal transfer function $H(s)$ that approximates $1/\sqrt{s}$ can be used to model 1/f noise[10]. White noise is generated using sum of sinusoids with random phases or piecewise constant pulses with random amplitudes[11]. Noises with power-law spectra such as f^{-3} and $1/f$ have unique, fundamental properties that these methods fail to address. First, the f^{-3} process has infinite memory (long-term correlation), and is nonstationary. This is not unlike the Brownian motion that characterizes the phase deviation in oscillators that are perturbed by white noise. Second, the f^{-3} process exhibits self-similar features, and is scale invariant. Changing measurement time or frequency scales does not alter the autocorrelation function or the spectral density function, except for a scaling of the magnitude by a multiplying constant. One method, commonly used in the field of computer graphics, is the random displacement algorithm[12]. We use an accurate method that was proposed by Hosking[13] and has the desired statistical properties. The f^{-3} process is simulated by convolving independent Gaussians with the discrete-time transfer function

$$H(z) = 1\big/(1 - z^{-1})^{3/2}.$$

To model nonstationary oscillator phase deviation with spectral density

$$Sf(f) = c\,\frac{f_0^2}{f^2} + c_{FN}\,\frac{f_0^2}{|f|^3},$$

we define the continuous-time process

$$s(t) = t + \sqrt{c}\int_0^t w(t)\,dt + \sqrt{c_{FN}}\int_0^t w_{FN}(t)\,dt,$$

where $w(t)$ is stationary white noise and $w_{FN}(t)$ is nonstationary $1/f$ noise. We simulate this continuous-time process with timestep Δt using the discrete-time process defined by

$$s_{k+1} = \sum_{n=0}^{k}(\Delta t + \sqrt{c\,\Delta t}\,w_n) + \sqrt{2\pi\,c_{FN}}\,\Delta t \sum_{n=0}^{k} h_{k-n} w'_n,$$

with w_n and w'_n as independent Gaussians. The last term has increments with standard deviation proportional to the time scale. Note that the convolution sum is expensive to evaluate directly. One way to compute the entire sequence of convolution sums is to apply FFT to h and w', multiply the two resulting vectors, and apply inverse FFT. It takes several CPU seconds to generate a 2^{17}-point noise sequence (with 131,072 samples), instead of one hour required by direct convolution in the time domain. The phase noise spectral density is given by

$$(2\pi\,f_0)^2\left\{\frac{c\,\Delta t^2}{(2\sin\pi\,f\,\Delta t)^2} + \frac{2\pi\,c_{FN}\,\Delta t^3}{|2\sin\pi\,f\,\Delta t|^3}\right\},$$

and this matches the desired spectrum for frequencies below the Nyquist frequency. As before, we simulate the random transition times of an oscillator using the s_k process with $\Delta t = T/2$, or $\Delta t = T/4$ for an oscillator with quadrature outputs. When a divide-by-R circuit follows the oscillator, we use the s_k process with $\Delta t = RT/2$ to generate the jittered transition times of the divider.

In Listing 1, we give a VHDL model of an oscillator-divider circuit. Stationary, independent jitter due to divider amplitude noise at the time of transition as well as jitter due to oscillator phase noise is modeled. We set the `Josc` generic to the rms cycle-to-cycle jitter $\sqrt{cT}$ due to white noise, with T as 1/`Freq` and c extracted from the f_m^{-2} region of the oscillator phase noise spectrum. The `CFNosc` generic is set to c_{FN} extracted from the f_m^{-3} region that is due to flicker noise. A value for the `Jdiv` generic can be determined from time-domain noise simulation of the divider. The model makes two calls to a standard Gaussian (normal) random number generator `rnorm` for each output transition. The `rv3` vector is computed by our fast f^{-3} random number generator `roscfn_vec` that is implemented in ADVance MS using C code encapsulation.

5. Experimental Results

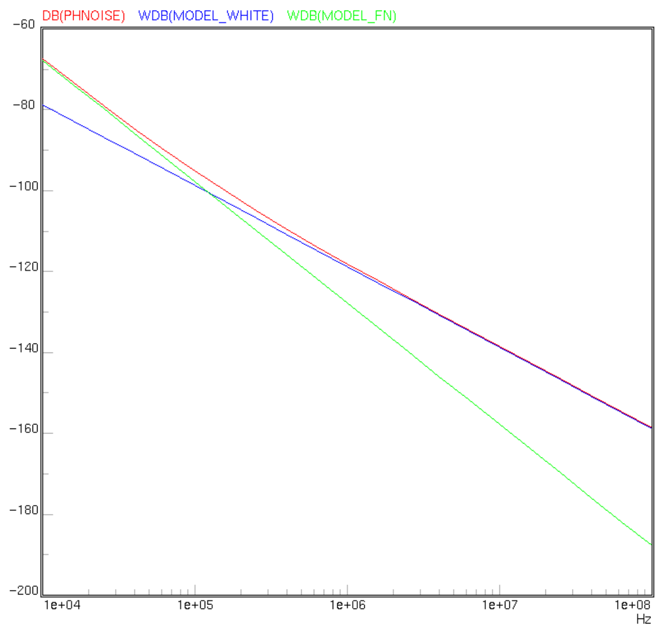
To verify this work, we simulated a 1.826 GHz CMOS oscillator circuit[14] using Eldo RF. The double sideband phase-noise spectrum, $2L(f_m)$, is shown in Figure 6, and is in good agreement with measurements. The effect of flicker noise becomes dominant at frequency offsets below 100 kHz. At an offset frequency f_m = 600 kHz, the measured and the simulated results are the same, $L(f_m)$ = -116 dBc/Hz. From the $2L(f_m) \approx 2 f_0^2 (c\,f_m^{-2} + c_{FN}\,f_m^{-3})$ spectrum, we extracted the two model parameters: $c = 2.0e-19$ and $c_{FN} = 2.5e-14$. The rms cycle-to-cycle jitter, due to white noise, is $\sqrt{c\,T}$ or 0.0019%. We then simulated the oscillator using our time-domain jitter model s_k. The resulting phase-noise spectrum, $Sf(f)$, is shown in Figure 7 and is in excellent agreement with the expected spectrum.

```
entity dosc_fd is
  generic (Freq  : real := 100.0e6;  -- oscillator frequency in Hz
           Ratio : real := 1.0;      -- divider ratio
           TD    : time := 0 sec;    -- output delay
           Jdiv  : time := 0 sec;    -- sigma(jitter at one divider transition)
           Josc  : time := 0 sec;    -- sigma(cycle-to-cycle oscillator jitter)
           CFNosc : real    := 0.0;  -- oscillator phase noise due to 1/f noise
           Kcycle : integer := 10;   -- # of oscillator-divider cycle = 2 ^ Kcycle
           Seed1 : integer := 0;
           Seed2 : integer := 129792743);
  port (DOUT : out bit := '0');
begin
  assert Freq  > 0.0;
  assert Ratio > 0.0;
end entity dosc_fd;

architecture bhv_jitter_fn of dosc_fd is
  constant halfPeriod : time := ( 0.5 * Ratio / Freq ) * sec;
begin
  process
   variable delta : time := 0 sec;
   variable state : bit  := '0';
   variable seed_1 : integer := Seed1;
   variable seed_2 : integer := Seed2;
   variable rn1 : real;
   variable rn2 : real;
   variable rv3 : real_vector( 1 to 2 ** (Kcycle+1) );
  begin
   roscfn_vec( seed_1, seed_2, rv3, halfPeriod );
   for i in rv3'range loop
     rnorm( seed_1, seed_2, rn1 );
     rnorm( seed_1, seed_2, rn2 );
     wait for halfPeriod + sqrt(Ratio/2.0) * Josc * rn1 + (TD + Jdiv * rn2 - delta) +
        0.5 * MATH_1_OVER_PI * sqrt(CFNosc) * rv3(i);
     state := not state;
     DOUT <= state;
     delta := TD + Jdiv * rn2;
   end loop;
   assert FALSE report "roscfn_vec data exhausted" severity ERROR;
  end process;
end bhv_jitter_fn;
```

Listing 1. VHDL model of oscillator-divider circuit with timing jitter

Figure 6. Double sideband phase-noise spectrum of CMOS oscillator circuit



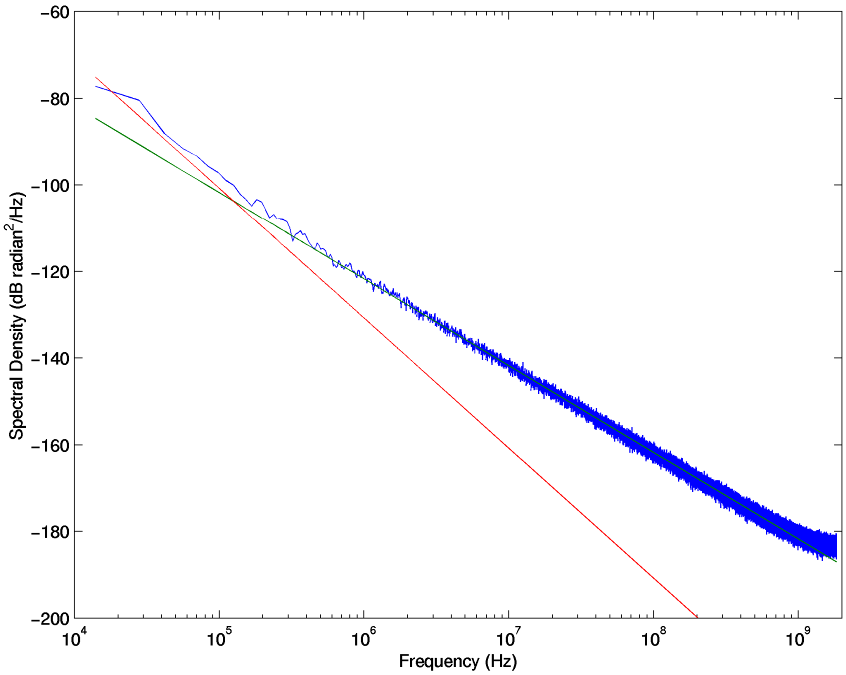


Figure 7. Phase noise spectrum of oscillator timing jitter model

6. Summary

In this paper, we reviewed a theory of phase noise in oscillators, and identified some common misconceptions in oscillator analyses. We linked oscillator phase noise spectrum to timing jitter, and derived timing jitter models suitable for discrete event simulation. Oscillator transition-time jitter caused by white noise is characterized by the inverse Gaussian probability density function. Oscillator phase jitter due to white noise and flicker noise has inherent nonstationary and self-similar properties, and these properties are preserved in our model. We presented simulation results for a CMOS oscillator circuit. Jitter simulation of oscillators, VCOs, frequency dividers, phase detectors, charge pumps and loop filters in complete phase-locked loops will be presented in a separate paper.

Acknowledgment

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[1] F.K. Kärtner, "Analysis of white and f^{-a} noise in oscillators," Int. J. Circuit Theory Appl., Vol. 18, pp. 485-519, 1990.

[2] A. Demir, A. Mehrotra and J. Roychowdhury, "Phase Noise in Oscillators: A Unifying Theory and Numerical Methods for Characterization," IEEE Trans. Circuits and Systems I, Vol. 47, No. 5, pp. 655-673, May 2000.

[3] In this paper, we make a careful distinction between stationary, cyclostationary and nonstationary processes. A stationary process has time-invariant statistics. A cyclostationary process exhibits periodically time-varying statistics. A nonstationary process has non-cyclic, time-dependent statistics.

[4] The power spectral density of a wide sense stationary process is the Fourier transform of the autocorrelation function. The power spectral density of a cyclostationary process can be defined in a similar fashion. However, for nonstationary processes such as a Wiener process, the power spectral density does not formally exist. Here, we use this term to refer to the spectrum obtained using a periodogram.

[5] A. Hajimiri and T. Lee, "A General Theory of Phase Noise in Electrical Oscillators," IEEE Journal of Solid-State Circuits, Vol. 33, No. 2, pp. 179-194, February 1998.

[6] K. Kundert, "Modeling and Simulation of Jitter in Phase-Locked Loops," in *Advances in Analog Circuit Design*, 1997.

[7] D.B. Leeson, "A Simple Model of Feedback Oscillator Noise Spectrum," Proceedings of the IEEE, Vol. 43, pp. 329-330, February 1966.

[8] Typically, the flat noise floor represents the average buffer noise over time and thus does not characterize jitter at the sampling instant when output transitions occur.

[9] P. Bolcato and R. Poujois, "A new approach for noise simulation in transient analysis," Proc. IEEE Int. Symp. on CAS, pp. 887-890, June 1992.

[10] A. Demir, E. Liu, and A. Sangiovanni-Vincentelli, "Time-Domain non-Monte Carlo Noise Simulation for Nonlinear Dynamic Circuits with Arbitrary Excitations," IEEE Trans. CAD, Vol. 15, No. 5, pp. 493-505, May 1996.

[11] J.A. McNeill, "Jitter in ring-oscillators," Ph.D. dissertation, Boston University, Boston, MA, 1994.

[12] A. Fournier, D. Fussell and L. Carpenter, "Computer Rendering of Stochastic Models," Communications of the ACM, Vol. 25, No. 6, pp. 371-384, 1982.

[13] J.R.M. Hosking, "Fractional differencing," Biometrika, Vol. 18, No. 1, pp. 165-176, 1981.

[14] J. Craninckx and M. Steyaert, "A 1.8GHz Low-Phase-Noise CMOS VCO using Optimized Hollow Spiral Inductors," IEEE Journal of Solid-State Circuits, Vol. 32, No. 5, May 1997.